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Convergence of a Generalized Pulse-Spectrum Technique (GPST) for Inverse Problems of 1-D Diffusion Equations in Space-Time Domain

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Abstract. The problem of convergence of a special form of the generalized pulse-spectrum technique (GPST) for solving inverse problems of one-dimensional diffusion equations in space-time domain is considered. Under the assumptions that a Tikhonov regularized solution exists and the derivative operator of the regularized forward problem at the regularized solution is invertible, the iterative solutions of this special GPST converge to the Tikhonov regularized solution in C norm if the initial guess is close enough to the Tikhonov regularized solution and the rate of convergence is at least linear.

1. Introduction. The generalized pulse-spectrum technique (GPST) [1] is a versatile and efficient iterative numerical algorithm for solving inverse problems of a system of nonlinear partial differential equations. In general, inverse problems of partial differential equations can be formulated as ill-posed nonlinear operator equations. It is important to point out that the GPST is not a single narrowly defined iterative numerical algorithm, but a broad class of iterative numerical algorithms based on the concept that either the nonlinear operator equation is first linearized by any one of the Newton-like iteration methods and then each iterate is solved by using a stabilizing method to overcome the instability, e.g., the Tikhonov regularization method [14]. Alternatively, the stabilizing method can be first applied to the nonlinear operator and then the stabilized nonlinear problem is solved by using a Newton-like iteration. Hence different choices of various Newton-like iteration methods and stabilizing methods lead to different special forms of GPST. The choice of a specific Newton-like iteration method and stabilizing method and the question of whether to solve the inverse problem in the space-time domain or in the space-complex frequency domain depend mainly on the particular inverse problem under consideration. The efficiency of a GPST depends upon how efficiently one can treat every single step in the particular numerical algorithm.

It has been demonstrated that the GPST iterative numerical algorithm does give very good results in solving the inverse problems with time-dependent coefficients of one-, two- and three-dimensional linear evolution partial differential equations in the space-complex frequency domain, [2]–[7], [10], [14], and in the space-time domain, [13]. Similarly, the inverse problems with time-dependent coefficients of a one-dimensional linear diffusion equation can be solved by using the GPST with equal efficacy [11]. The convergence of a special form of GPST for solving inverse

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problems of one-dimensional evolutional partial differential equations in the space-complex frequency domain has been proved under several assumptions in [15].

In Section 2 a special form of GPST for solving one-dimensional diffusion equations in the space-time domain is introduced. This is followed by a section on the mathematical properties of the nonlinear inverse operator A from $C[0,1]$ to $C[0,T]$. Finally, in Section 4 the convergence proof and error estimates are given. They show that if the initial guess $k_0(x)$ is close enough to the regularized solution $k_\alpha(x)$, then under the assumptions that a Tikhonov regularized solution exists and the derivative operator of the regularized forward problem at the regularized solution is invertible, the iterative sequence $\{k_n(x)\}$, $n = 0, 1, 2, 3, \ldots$, of this special form of GPST converges to $k_\alpha(x)$ and $||k_\alpha - k_n||_C < \text{Const}\cdot\varsigma^n$, $0 < \varsigma < 1$. The proof of the invertibility of the derivative operator of the regularized forward problem at the regularized solution will be presented elsewhere. It is clear that the generalization of this proof to cases of the corresponding inverse problems of higher spatial dimensions is rather straightforward but very tedious.

2. Generalized Pulse-Spectrum Technique (GPST). Consider the initial-boundary value problem of the one-dimensional linear diffusion equation

$$\tag{2.1} \begin{array}{ll} \partial(k(x)\partial u/\partial x)/\partial x - \partial u/\partial t = 0, & (x,t) \in Q = \{x \in (0,1),\ t \in (0,T)\}, \\ u(x,0) = 0, & 0 \le x \le 1, \\ \partial u(0,t)/\partial x = f(t), & 0 \le t \le T, \\ u(1,t) = 0, & 0 \le t \le T, \end{array}$$

and the auxiliary condition,

$$\tag{2.2} u(0,t) = g(t), \qquad 0 \le t \le T,$$

where $k(x) \in \Sigma = \{k(x) | k(x) \in C[0,1],\ k_- < k(x) < k_+ \text{ on } [0,1]\}$ with constants $k_-, k_+ > 0$ and $k_- \le 1$, $f(t) \in H^1(0,T)$, $f(0) = 0$ and $g(t) \in C[0,T]$, $g(0) = 0$, a function obtained from measurements.

The inverse problem here is to determine $k(x)$ such that $u(x,t)$ satisfies (2.1) and (2.2). Mathematically, let there exist a nonlinear operator A_1 mapping $k(x) \to u(x,t)$ and a trace operator A_2 mapping $u(x,t) \to g(t)$ on the proper part of the boundary. Hence the inverse problem amounts to solving the nonlinear operator equation,

$$\tag{2.3} A \cdot k(x) \equiv A_2 \cdot A_1 \cdot k(x) = g(t).$$

Mathematically, linearizing (2.3) by a Newton-like iteration method first, and then solving each iterate by the Tikhonov regularization method, is similar to first applying the Tikhonov regularization method to (2.3) and then solving its corresponding Euler equation by a Newton-like iteration method. In actual computation, the first approach is more straightforward, but for theoretical analysis the second approach seems to be more convenient.

Based upon the second approach, the Tikhonov regularization method for solving (2.3) is to minimize the functional

$$\tag{2.4} J_\alpha(k) \equiv ||A \cdot k - g||^2_{L^2(0,T)} + \alpha^2 \langle B \cdot k, k \rangle_{L^2[0,1]},$$

where $k(x) \in \Sigma$, B is a selfadjoint strictly positive bounded linear operator from $C[0,1]$ into its dual V_0 (the set of all regular functions of bounded variation on

$x \in [0,1]$ vanishing at $x = 0$), and α^2 is the regularization parameter. The Euler equation of (2.4) is

$$\phi(k) \equiv A'^*(k) \cdot (A \cdot k - g) + \alpha^2 B \cdot k = 0, \tag{2.5}$$

where $A'(k)$ is the Fréchet derivative of A at k and $A'^*(k)$ is the adjoint of $A'(k)$.

For the special form of GPST here, the following Newton-like iteration method,

$$k_{n+1} = k_n - N_n^{-1}[A'^*(k_n) \cdot (A \cdot k_n - g) + \alpha^2 B \cdot k_n], \qquad n = 0,1,2,3,\ldots, \tag{2.6}$$

with $N_n \equiv A'^*(k_n) \cdot A'(k_n) + A''^*(k_0) \cdot (A \cdot k_0 - g) + \alpha^2 B$ is used to solve the Euler equation (2.5).

3. Mathematical Properties of the Operator A. First, a basic lemma containing inequalities for integrals of the solution and its derivatives of an initial-boundary value problem of a parabolic equation is proved. Next, the existence of a unique continuous solution of (2.1) for any $k(x) \in \Sigma$ and $f(t) \in H^1(0,T)$, $f(0) = 0$, is also proved, i.e., the operator A is well defined. Finally, the Fréchet derivatives of A of arbitrary order are obtained as the solutions of various initial-boundary value problems of parabolic equations.

Consider the set of functions $\Sigma_\infty = \Sigma \cap C^\infty[0,1]$, which is a dense subset of Σ in the $C[0,1]$-norm, and the set $C^{\infty*}[0,T] = \{f(t) | f(t) \in C^\infty[0,T],\ f(0) = f^{(l)}(0) = 0,\ l = 1,2,\ldots\}$, which is a dense subset of $H^{1*}(0,T) = \{f(t) | f(t) \in H^1(0,T),\ f(0) = 0\}$ in the $H^1(0,T)$-norm.

LEMMA 1. *Let*

(i) $a(x) \in \Sigma_\infty$, $a_i(x) \in C^\infty[0,1]$, $i = 1,2,\ldots,m$,

(ii) $b(t) \in C^{\infty*}[0,T]$, *and*

(iii) $\psi_i(x,t) \in C^\infty(\mathbf{Q})$, $\psi_i(x,0) = 0$, $\partial\psi_i(0,t)/\partial x = c_i(t) \in C^{\infty*}[0,T]$, $i = 1,\ldots,m$.

Then there exists a unique $C^\infty(\mathbf{Q})$ solution $\theta(x,t)$ of the following initial-boundary value problem:

$$\begin{aligned} &\partial\theta/\partial t - \partial(a(x)\partial\theta/\partial x)/\partial x = \partial\left(\sum_{i=1}^{m} a_i(x)\partial\psi_i/\partial x\right)/\partial x, && (x,t) \in Q,\\ &\theta(x,0) = 0, && 0 \le x \le 1,\\ &\partial\theta(0,t)/\partial x = b(t), && 0 \le t \le T,\\ &\theta(1,t) = 0, && 0 \le t \le T. \end{aligned} \tag{3.1}$$

Moreover, $\theta(x,t)$ satisfies

$$||\partial\theta/\partial x||_{2,Q_\tau} \le M_0, \tag{3.2}$$

$$|\partial\theta/\partial t|_{2,\tau} \quad \textit{and} \quad ||\partial^2\theta/\partial t\partial x||_{2,Q_\tau} \le M_1, \tag{3.3}$$

$$|\partial\theta/\partial x|_{2,\tau} \quad \textit{and} \quad ||\theta||_{C(\mathbf{Q})} \le (2M_0M_1)^{1/2} \tag{3.4}$$

for any $\tau \in [0,T]$, where

$$|a_i|_0 = \operatorname*{Max}_{x\in[0,1]} |a_i(x)|,$$

$$|\theta|^2_{2,\tau} = \int_0^1 \theta^2(x,\tau)\,dx, \qquad ||\theta||^2_{2,Q_\tau} = \int_0^\tau \int_0^1 \theta^2(x,t)\,dx\,dt,$$

$$M_0 = k_-^{-1}\left\{k_+||b||_{L^2(0,T)} + \sum_{i=1}^m |a_i|_0(||c_i||_{L^2(0,T)} + ||\partial\psi_i/\partial x||_{2,Q_\tau})\right\},$$

and

$$M_1 = k_-^{-1}\left\{k_+||b'||_{L^2(0,T)} + \sum_{i=1}^m |a_i|_0(||c_i'||_{L^2(0,T)} + ||\partial^2\psi_i/\partial t\partial x||_{2,Q_\tau})\right\}.$$

Proof. The proof of the existence of a $C^\infty(\mathbf{Q})$ solution of (3.1) can be found in [8]. To prove (3.2), multiply the first equation of (3.1) by $\theta(x,t)$ and integrate by parts on $\mathbf{Q}_\tau = \mathbf{Q} \cap [0,\tau]$ for $\tau \in [0,T]$, to obtain

$$\begin{aligned}
&\tfrac{1}{2}|\theta|_{2,\tau}^2 + ||a^{1/2}\partial\theta/\partial x||_{2,Q_\tau}^2 \\
&\quad = -\int_0^\tau \left[a(0)b(t) + \sum_{i=1}^m a_i(0)c_i(t)\right]\theta(0,t)\,dt \\
&\qquad - \int_0^\tau\int_0^1 \sum_{i=1}^m a_i(x)\partial\psi_i/\partial x \cdot \partial\theta/\partial x\,dx\,dt \\
&\quad = \int_0^\tau\int_0^1 \left[a(0)b(t) + \sum_{i=1}^m a_i(0)c_i(t) - \sum_{i=1}^m a_i(x)\partial\psi_i/\partial x\right]\partial\theta/\partial x\,dx\,dt \\
&\quad \le \tfrac{1}{2}||a^{1/2}\partial\theta/\partial x||_{2,Q_\tau}^2 \\
&\qquad + \tfrac{1}{2}\int_0^\tau\int_0^1 a^{-1}(x)\left[a(0)b(t) + \sum_{i=1}^m a_i(0)c_i(t) - \sum_{i=1}^m a_i(x)\partial\psi_i/\partial x\right]^2 dx\,dt.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&|\theta|_{2,\tau}^2 + ||a^{1/2}\partial\theta/\partial x||_{2,Q_\tau}^2 \\
&\quad \le k_-^{-1}\left\{k_+||b||_{L^2(0,T)} + \sum_{i=1}^m |a_i|_0(||c_i||_{L^2(0,T)} + ||\partial\psi_i/\partial x||_{2,Q_\tau})\right\}^2.
\end{aligned}$$

Inequality (3.2) now follows.

Let $\sigma(x,t) = \partial\theta(x,t)/\partial t$. This satisfies the initial-boundary value problem,

$$(3.5)\quad \begin{aligned}
&\partial\sigma/\partial t - \partial(a(x)\partial\sigma/\partial x)/\partial x = \partial\left(\sum_{i=1}^m a_i(x)\partial^2\psi_i/\partial t\partial x\right)/\partial x, \qquad (x,t)\in Q,\\
&\sigma(x,0) = \partial\left(a(x)\partial\theta/\partial x + \sum_{i=1}^m a_i(x)\partial\psi_i/\partial x\right)/\partial x|_{t=0} = 0, \qquad 0\le x\le 1,\\
&\partial\sigma(0,t)/\partial x = b'(t), \qquad 0\le t\le T, \text{ and}\\
&\sigma(1,t) = 0, \qquad 0\le t\le T.
\end{aligned}$$

Upon comparing (3.1) with (3.5) one concludes that $\sigma(x,t)$ satisfies the estimation

$$\begin{aligned}
&|\sigma|_{2,\tau}^2 + ||a^{1/2}\partial\sigma/\partial x||_{2,Q_\tau}^2 \\
&\quad \le k_-^{-1}\left\{k_+||b'||_{L^2(0,T)} + \sum_{i=1}^m |a_i|_0(||c'||_{L^2(0,T)} + ||\partial^2\psi_i/\partial t\partial x||_{2,Q_\tau})\right\}^2 \\
&\quad = k_-M_1^2.
\end{aligned}$$

Hence (3.3) is true. Finally, multiplying the first equation of (3.1) by $\partial\theta(x,t)/\partial t$, integrating by parts on $\mathbf{Q}_\tau$, and following a similar procedure, one obtains (3.4). □

LEMMA 2. *For any $k(x) \in \Sigma_\infty$ and $f(t) \in C^{\infty *}[0,T]$ there exists a unique $C^\infty(\mathbf{Q})$ solution $u(x,t)$ of* (2.1) *which satisfies*

$$\|\partial u/\partial x\|_{2,Q_\tau} \le k_+ k_-^{-1} \|f\|_{L^2(0,T)}, \tag{3.6}$$

$$|\partial u/\partial t|_{2,\tau} \text{ and } \|\partial^2 u/\partial t \partial x\|_{2,Q_\tau} \le k_+ k_-^{-1} \|f'\|_{L^2(0,T)}, \tag{3.7}$$

$$|\partial u/\partial x|_{2,\tau} \text{ and } \|u\|_{C(\mathbf{Q})} \le 2^{1/2} k_+ k_-^{-1} \|f\|_{H^1(0,T)} \tag{3.8}$$

for any $\tau \in [0,T]$. Moreover, for any $k^*(x)$, $k(x) \in \Sigma_\infty$ and $f^*(t)$, $f(t) \in C^{\infty *}[0,T]$, denote by $u^*(x,t)$ and $u(x,t)$ the solutions of (2.1) *corresponding to $\{k^*, f^*\}$ and $\{k, f\}$, respectively; then their difference $u^*(x,t) - u(x,t)$ satisfies*

$$\|\partial(u^* - u)/\partial x\|_{2,Q_\tau} \le \frac{k_- + k_+}{k_-^2} \|f^*\|_{L^2(0,T)} |k^* - k|_0 + k_+ k_-^{-1} \|f^* - f\|_{L^2(0,T)}, \tag{3.9}$$

$$\begin{aligned} &|\partial(u^* - u)/\partial t|_{2,\tau} \text{ and } \|\partial^2(u^* - u)/\partial t \partial x\|_{2,Q_\tau} \\ &\quad \le \frac{k_- + k_+}{k_-^2} \|f^{*\prime}\|_{L^2(0,T)} |k^* - k|_0 + k_+ k_-^{-1} \|f^{*\prime} - f'\|_{L^2(0,T)}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &|\partial(u^* - u)/\partial x|_{2,\tau} \text{ and } \|u^* - u\|_{C(\mathbf{Q})} \\ &\quad \le 2^{1/2} \left[\frac{k_- + k_+}{k_-^2} \|f^*\|_{H^1(0,T)} |k^* - k|_0 + k_+ k_-^{-1} \|f^* - f\|_{H^1(0,T)} \right] \end{aligned} \tag{3.11}$$

for any $\tau \in [0,T]$.

Proof. This lemma can be proved by using Lemma 1 for the direct problem (2.1) and the initial-boundary value problem satisfied by the difference $u^*(x,t) - u(x,t)$. □

LEMMA 3. *For any $k(x) \in \Sigma_\infty$, $h_i(x) \in C^\infty[0,1]$, $i = 1, 2, 3, \ldots, f(t) \in C^{\infty *}[0,T]$, and any positive integer l, there exists a unique $C^\infty(\mathbf{Q})$ solution $w_{1,2,\ldots,l}(x,t)$ of the initial-boundary value problem,*

$$\begin{aligned} &\partial w_{1,2,\ldots,l}/\partial t - \partial(k(x) \partial w_{1,2,\ldots,l}/\partial x)/\partial x \\ &\quad = \partial \left(\sum_{i=1}^{l} h_i(x) \partial w_{1,2,\ldots,\not{i},\ldots,l}/\partial x \right) /\partial x, \qquad (x,t) \in Q, \end{aligned} \tag{3.12}$$

$$w_{1,2,\ldots,l}(x,0) = 0, \qquad 0 \le x \le 1,$$

$$\partial w_{1,2,\ldots,l}(0,t)/\partial x = 0, \quad w_{1,2,\ldots,l}(1,t) = 0, \qquad 0 \le t \le T,$$

where $w_{1,2,\ldots,\not{i},\ldots,l} \equiv w_{1,2,\ldots,i-1,i+1,\ldots,l}$, $w_{\not{1}}(0,t) = u(0,t)$ and $\partial w_{\not{1}}(0,t)/\partial x = f(t)$. $w_{1,2,\ldots,l}(x,t)$ satisfies

$$\|\partial w_{1,2,\ldots,l}/\partial x\|_{2,Q_\tau} \le \frac{l!(k_- + k_+)}{k_-^{l+1}} \|f\|_{L^2(0,T)} |h_1|_0 |h_2|_0 \cdots |h_l|_0, \tag{3.13}$$

$$\begin{aligned} &|\partial w_{1,2,\ldots,l}/\partial t|_{2,\tau} \text{ and } \|\partial^2 w_{1,2,\ldots,l}/\partial t \partial x\|_{2,Q_\tau} \\ &\quad \le \frac{l!(k_- + k_+)}{k_-^{l+1}} \|f'\|_{L^2(0,T)} |h_1|_0 |h_2|_0 \cdots |h_l|_0, \end{aligned} \tag{3.14}$$

$$(3.15)\qquad \begin{aligned}&|\partial w_{1,2,\ldots,l}/\partial x|_{2,\tau} \text{ and } \|w_{1,2,\ldots,l}\|_{C(\mathbf{Q})}\\ &\le \frac{2^{1/2} l!(k_- + k_+)}{k_-^{l+1}} \|f\|_{H^1(0,T)} |h_1|_0 |h_2|_0 \cdots |h_l|_0,\end{aligned}$$

for any $\tau \in [0,T]$ *and* $l = 1,2,3,\ldots$. *Moreover, for any* $k^*(x)$ *and* $k(x) \in \Sigma_\infty$, $h_i^*(x)$ *and* $h_i(x) \in C^\infty[0,1]$, $i = 1,2,\ldots$, $f^*(t)$ *and* $f(t) \in C^{\infty *}[0,T]$, *and any positive integer* l, *denote by* $w_{1,2,\ldots,l}^*(x,t)$ *and* $w_{1,2,\ldots,l}(x,t)$ *the solutions of* (3.12) *corresponding to* $\{k^*, h_1^*, h_2^*, \ldots, h_l^*, f^*\}$ *and* $\{k, h_1, h_2, \ldots, h_l, f\}$, *respectively; then their difference* $w_{1,2,\ldots,l}^*(x,t) - w_{1,2,\ldots,l}(x,t)$ *satisfies*

$$(3.16)\qquad \begin{aligned}&\|\partial(w_{1,\ldots,l}^* - w_{1,\ldots,l})/\partial x\|_{2,Q_\tau}\\ &\le \frac{(l+1)!(k_- + k_+)}{k_-^{l+2}} F_0 H_1 H_2 \cdots H_l\\ &\quad \cdot \left\{ |k^* - k|_0 + \frac{k_-}{l+1} \left(\sum_{i=1}^{l} \frac{|h_i^* - h_i|_0}{H_i} + \frac{\|f^* - f\|_{L^2(0,T)}}{F_0} \right) \right\},\end{aligned}$$

$$(3.17)\qquad \begin{aligned}&|\partial(w_{1,\ldots,l}^* - w_{1,\ldots,l})/\partial t|_{2,\tau} \text{ and } \|\partial^2(w_{1,\ldots,l}^* - w_{1,\ldots,l})/\partial t \partial x\|_{2,Q_\tau}\\ &\le \frac{(l+1)!(k_- + k_+)}{k_-^{l+2}} F_1 H_1 H_2 \cdots H_l\\ &\quad \cdot \left\{ |k^* - k|_0 + \frac{k_-}{l+1} \left(\sum_{i=1}^{l} \frac{|h_i^* - h_i|_0}{H_i} + \frac{\|f^{*\prime} - f'\|_{L^2(0,T)}}{F_1} \right) \right\},\end{aligned}$$

$$(3.18)\qquad \begin{aligned}&|\partial(w_{1,\ldots,l}^* - w_{1,\ldots,l})/\partial x|_{2,\tau} \text{ and } \|w_{1,\ldots,l}^* - w_{1,\ldots,l}\|_{C(\mathbf{Q})}\\ &\le \frac{2^{1/2}(l+1)!(k_- + k_+)}{k_-^{l+2}} (F_0 + F_1) H_1 \cdots H_l\\ &\quad \cdot \left\{ |k^* - k|_0 + \frac{k_-}{l+1} \left(\sum_{i=1}^{l} \frac{|h_i^* - h_i|_0}{H_i} + \frac{\|f^* - f\|_{H^1(0,T)}}{F_0 + F_1} \right) \right\}\end{aligned}$$

for any $\tau \in [0,T]$ *and* $l = 1,2,3,\ldots$. *Here the positive constants* $F_0, F_1, H_1, H_2, \ldots$ *are chosen such that*

$$(3.19)\qquad \begin{aligned}&\mathrm{Max}\{\|f^*\|_{L^2(0,T)}, \|f\|_{L^2(0,t)}\} \le F_0,\\ &\mathrm{Max}\{\|f^{*\prime}\|_{L^2(0,T)}, \|f'\|_{L^2(0,T)}\} \le F_1,\\ &\mathrm{Max}\{|h_i^*|_0, |h_i|_0\} \le H_i, \qquad i = 1,2,3,\ldots.\end{aligned}$$

Proof. Applying Lemma 1 and Lemma 2 to (3.12) and to the initial-boundary value problem for $w_{1,2,\ldots,l}^*(x,t) - w_{1,2,\ldots,l}(x,t)$, and using induction on l, the desired inequalities follow after some tedious manipulation. □

Remark. For the special case of $h_i^*(x) = h_i(x)$, $i = 1,2,3,\ldots$, and $f^*(t) = f(t)$, Eqs. (3.16), (3.17) and (3.18) become, respectively,

$$(3.20)\qquad \begin{aligned}&\|\partial(w_{1,\ldots,l}^* - w_{1,\ldots,l})/\partial x\|_{2,Q_\tau}\\ &\le \frac{(l+1)!(k_- + k_+)}{k_-^{l+2}} \|f\|_{L^2(0,T)} |h_1|_0 \cdots |h_l|_0 |k^* - k|_0,\end{aligned}$$

$$(3.21)\qquad \begin{aligned}&|\partial(w_{1,\ldots,l}^* - w_{1,\ldots,l})/\partial t|_{2,\tau} \text{ and } \|\partial^2(w_{1,\ldots,l}^* - w_{1,\ldots,l})/\partial t \partial x\|_{2,Q_\tau}\\ &\le \frac{(l+1)!(k_- + k_+)}{k_-^{l+2}} \|f'\|_{L^2(0,T)} |h_1|_0 |h_2|_0 \cdots |h_l|_0 |k^* - k|_0,\end{aligned}$$

$$|\partial(w^*_{1,\dots,l} - w_{1,\dots,l})/\partial x|_{2,\tau} \text{ and } ||w^*_{1,\dots,l} - w_{1,\dots,l}||_{C(\mathbf{Q})} \tag{3.22}$$
$$\le \frac{2^{1/2}(l+1)!(k_- + k_+)}{k_-^{l+2}} ||f||_{H^1(0,T)} |h_1|_0 |h_2|_0 \cdots |h_l|_0 |k^* - k|_0.$$

LEMMA 4. *For any* $k^*(x)$, $k(x) \in \Sigma_\infty$, $h_i(x) \in C^\infty[0,1]$, $i = 1, 2, \dots,$ *and* $f(t) \in C^{\infty *}[0,T]$, *the differences*

$$p_1(x,t) = u^*(x,t) - u(x,t) - w_1(x,t) \quad \text{and}$$
$$p_{1,2,\dots,l}(x,t) = w^*_{1,\dots,l-1}(x,t) - w_{1,\dots,l-1}(x,t) - w_{1,\dots,l}(x,t), \tag{3.23}$$
$$l = 2, 3, \dots,$$

satisfy

$$||p_1||_{C(\mathbf{Q})} \le \frac{2^{1/2}(k_- + k_+)}{k_-^3} ||f||_{H^1(0,T)} |k^* - k|_0^2 \tag{3.24}$$

and

$$||p_{1,\dots,l}||_{C(\mathbf{Q})} \le \frac{2^{1/2}(l+1)!(k_- + k_+)}{k_-^{l+2}} ||f||_{H^1(0,T)} |h_1|_0 \cdots |h_{l-1}|_0 |k^* - k|_0^2, \tag{3.25}$$
$$l = 2, 3, 4, \dots.$$

Here, $w^*_{1,2,\dots,l-1}(x,t)$, $w_{1,2,\dots,l-1}(x,t)$ *and* $w_{1,2,\dots,l}(x,t)$ *are the solutions of* (3.12) *corresponding to* $\{k^*, h_1, \dots, h_{l-1}, f\}$, $\{k, h_1, \dots, h_{l-1}, f\}$ *and* $\{k, h_1, \dots, h_{l-1}, k^* - k, f\}$, *respectively.*

Proof. This lemma is a direct consequence of Lemma 1 and the remark after Lemma 3. □

LEMMA 5. *For all* $k(x) \in \Sigma$ *one has*

$$A \cdot k(x) \subset C[0,T]. \tag{3.26}$$

Proof. For any fixed $k(x) \in \Sigma$ and $f(t) \in H^{1*}(0,T)$ one can construct their approximate sequences $\{k_n(x)\} \subset \Sigma_\infty$ and $\{f_n(t)\} \subset C^{\infty *}[0,T]$ such that

$$|k_n - k|_0 \to 0, \qquad ||f_n - f||_{H^1(0,T)} \to 0 \quad \text{as } n \to \infty.$$

Without loss of generality, let

$$||f_n||_{L^2(0,T)} \le ||f||_{L^2(0,T)} + 1 \equiv F_0 \quad \text{and} \quad ||f'_n||_{L^2(0,T)} \le ||f'||_{L^2(0,T)} + 1 \equiv F_1.$$

Consider the family of initial-boundary value problems

$$\begin{aligned}
&\partial u_n/\partial t - \partial(k_n(x)\partial u_n/\partial x)/\partial x = 0, \qquad (x,t) \in Q,\\
&u_n(x,0) = 0, \qquad 0 \le x \le 1,\\
&\partial u_n(0,t)/\partial x = f_n(t), \qquad 0 \le t \le T,\\
&u_n(1,t) = 0, \qquad 0 \le t \le T.
\end{aligned}$$

Using the results of Lemma 1 for the $C^\infty(\mathbf{Q})$ solution of the above initial-boundary value problems and their differences $u_n(x,t) - u_m(x,t)$, $m, n = 1, 2, \dots,$ one can derive the following estimates,

$$\|\partial u_n/\partial x\|_{2,Q_\tau} \le k_+k_-^{-1}\|f_n\|_{L^2(0,T)} \le k_+k_-^{-1}F_0,$$
$$|\partial u_n/\partial t|_{2,\tau} \text{ and } \|\partial^2 u_n/\partial t\partial x\|_{2,Q_\tau} \le k_+k_-^{-1}\|f_n'\|_{L^2(0,T)} \le k_+k_-^{-1}F_1,$$
$$|\partial u_n/\partial x|_{2,\tau} \text{ and } \|u_n\|_{C(\mathbf{Q})} \le 2^{1/2}k_+k_-^{-1}\|f_n\|_{H^1(0,T)} \le 2^{1/2}k_+k_-^{-1}(F_0+F_1),$$
$$\|\partial(u_n-u_m)/\partial x\|_{2,Q_\tau} \le (k_-+k_+)k_-^{-2}F_0|k_n-k_m|_0 + k_+k_-^{-1}\|f_n-f_m\|_{L^2(0,T)},$$
$$|\partial(u_n-u_m)/\partial t|_{2,\tau} \;\&\; \|\partial^2(u_n-u_m)/\partial t\partial x\|_{2,Q_\tau}$$
$$\le \frac{k_-+k_+}{k_-^2}F_1|k_n-k_m|_0 + \frac{k_+}{k_-}\|f_n'-f_m'\|_{L^2(0,T)},$$

and

$$|\partial(u_n-u_m)/\partial x|_{2,\tau} \;\&\; \|u_n-u_m\|_{C(\mathbf{Q})}$$
$$\le 2^{1/2}\left\{\frac{k_-+k_+}{k_-^2}(F_0+F_1)|k_n-k_m|_0 + \frac{k_+}{k_-}\|f_n-f_m\|_{H^1(0,T)}\right\}.$$

From the above estimates it follows that there exists a function $u(x,t) \in C(\mathbf{Q})$ such that

$$u_n(x,t) \xrightarrow{C(\mathbf{Q})} u(x,t),$$

$$\partial u_n(x,t)/\partial x \xrightarrow{L^2(0,1)} \partial u(x,t)/\partial x, \qquad \partial u_n(x,t)/\partial t \xrightarrow{L^2(0,1)} \partial u(x,t)/\partial t$$

for any $t \in |0,T|$, and

$$\partial^2 u_n(x,t)/\partial t\partial x \xrightarrow{L^2(\mathbf{Q})} \partial^2 u(x,t)/\partial t\partial x.$$

Moreover, the norms of $u(x,t)$ and its derivatives also satisfy the above estimates.

Let $V = \{v(x,t)|v(x,t) \in C(\mathbf{Q}),\ \partial v/\partial x \in L^2(Q)$, and $v(0,t)=0$ for $t \in [0,T]\}$. It can be easily verified that $u_n(x,t)$ satisfies the integral identity

$$\int_0^\tau\int_0^1 \{\partial u_n/\partial t\cdot v + k_n\partial u_n/\partial x\cdot\partial v/\partial x\}\,dx\,dt + \int_0^\tau k_n(0)f_n(t)v(0,t)\,dt = 0$$

for any $v \in V$ and $\tau \in [0,T]$. Therefore, $u(x,t)$ satisfies the same integral identity with $n \to \infty$. This implies that $u(x,t)$ is a continuous solution of (2.1) in the weak sense. From the uniqueness properties of weak solutions of linear parabolic equations, $u(x,t)$ is also a unique continuous solution of (2.1). Thus, for any $k(x) \in \Sigma$ and $f(t) \in H^{1*}(0,T)$, one has $A_1\cdot k(x) = u(x,t) \in C(\mathbf{Q})$ and $A\cdot k(x) = u(0,t) \in C[0,T]$. □

Remark. It is obviously true that $u(0,t) = -\int_0^1 \partial u(x,t)/\partial x\,dx$ for $t \in [0,T]$.

The following relation then exists between the operators A_1 and A:

$$A\cdot k(x) = -\int_0^1 \partial(A_1\cdot k(x))/\partial x\,dx \quad \text{for } t \in [0,T]. \tag{3.27}$$

LEMMA 6. *The operator A is Fréchet differentiable up to an arbitrary order and the lth-order Fréchet differential*

$$A^{(l)}(k)\cdot h_1h_2\cdots h_l = w_{1,2,\ldots,l}(0,t) \in C[0,T] \tag{3.28}$$

for any $k \in \Sigma$, $h_i \in C[0,1]$, $i = 1, 2, \ldots, l$, $l = 1, 2, \ldots$, and $f(t) \in H^{1*}(0,T)$, where $w_{1,2,\ldots,l}(x,t)$ is the continuous solution of the initial-boundary value problem (3.12) *corresponding to $\{k, h_1, h_2, \ldots, h_l, f\}$.*

Proof. Let $k(x) \in \Sigma$, $h_i(x) \in C[0,1]$, $i = 1, 2, \ldots$, and $f(t) \in H^{1*}(0,T)$ be given. Let the approximate sequences $\{k_n(x)\} \subset \Sigma_\infty$, $\{h_{in}(x)\} \subset C^\infty[0,1]$, $i = 1, 2, \ldots$, and $\{f_n(t)\} \subset C^{\infty *}[0,T]$ satisfy the conditions in the proof of Lemma 5: $|h_{in} - h_i|_0 \to 0$ as $n \to \infty$ and $|h_{in}|_0 \le |h_i|_0 + 1 \equiv H_i$, $i, n = 1, 2, \ldots$. Then one has the $C^\infty(\mathbf{Q})$ sequence solution $\{u_n(x,t)\}$ of (2.1) and the $C^\infty(\mathbf{Q})$ sequence solutions $\{w_{1,2,\ldots,l,n}(x,t)\}$ of (3.12), $l = 1, 2, 3, \ldots$. From Lemma 5, the limit function $u(x,t)$ is the continuous solution of (2.1), i.e., $A_1 \cdot k(x) = u(x,t)$ for any $k(x) \in \Sigma$. Using the results of Lemma 2 for $\{w_{1,2,\ldots,l,n}(x,t)\}$, one obtains the following estimates:

$$
\begin{aligned}
&||\partial w_{1,\ldots,l,n}/\partial x||_{2,Q_\tau} \le \frac{l!(k_- + k_+)}{k_-^{l+1}} F_0 H_1 H_2 \cdots H_l,\\
&|\partial w_{1,\ldots,l,n}/\partial t|_{2,\tau} \text{ and } ||\partial^2 w_{1,\ldots,l,n}/\partial t \partial x||_{2,Q_\tau}\\
&\qquad \le \frac{l!(k_- + k_+)}{k_-^{l+1}} F_1 H_1 H_2 \cdots H_l,\\
&|\partial w_{1,\ldots,l,n}/\partial x|_{2,\tau} \text{ and } ||w_{1,\ldots,l,n}||_{C(\mathbf{Q})}\\
&\qquad \le \frac{2^{1/2} l!(k_- + k_+)}{k_-^{l+1}} (F_0 + F_1) H_1 H_2 \cdots H_l.
\end{aligned}
\tag{3.29}
$$

Moreover,

$$
\begin{aligned}
&||\partial(w_{1,\ldots,l,n} - w_{1,2,\ldots,l,m})/\partial x||_{2,Q_\tau}\\
&\quad \le \frac{(l+1)!(k_- + k_+)}{k_-^{l+2}} F_0 H_1 H_2 \cdots H_l\\
&\quad \cdot \left\{ |k_n - k_m|_0 + \frac{k_-}{l+1} \left(\sum_{i=1}^{l} \frac{|h_{in} - h_{im}|_0}{H_i} + \frac{||f_n - f_m||_{L^2(0,T)}}{F_0} \right) \right\},\\
&|\partial(w_{1,\ldots,l,n} - w_{1,\ldots,l,m})/\partial t|_{2,\tau} \text{ and } ||\partial^2(w_{1,\ldots,l,n} - w_{1,\ldots,l,m})/\partial t \partial x||_{2,Q_\tau}\\
&\quad \le \frac{(l+1)!(k_- + k_+)}{k_-^{l+2}} F_1 H_1 \cdots H_l\\
&\quad \cdot \left\{ |k_n - k_m|_0 + \frac{k_-}{l+1} \left(\sum_{i=1}^{l} \frac{|h_{in} - h_{im}|_0}{H_i} + \frac{||f_n' - f_m'||_{L^2(0,T)}}{F_1} \right) \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\partial(w_{1,\ldots,l,n} - w_{1,\ldots,l,m})/\partial x|_{2,\tau} \text{ and } ||w_{1,\ldots,l,n} - w_{1,\ldots,l,m}||_{C(\mathbf{Q})}\\
&\quad \le \frac{2^{1/2}(l+1)!(k_- + k_+)}{k_-^{l+2}} (F_0 + F_1) H_1 \cdots H_l\\
&\quad \cdot \left\{ |k_n - k_m|_0 + \frac{k_-}{l+1} \left(\sum_{i=1}^{l} \frac{|h_{in} - h_{im}|_0}{H_i} + \frac{||f_n - f_m||_{H^2(0,T)}}{F_0 + F_1} \right) \right\}
\end{aligned}
$$

for any $\tau \in [0,T]$ and $m, n, l = 1, 2, \ldots$.

From the last three estimates it follows that there exist functions $w_{1,2,\dots,l}(x,t) \in C(\mathbf{Q})$, $l = 1, 2, 3, \dots$, such that

$$\begin{aligned} w_{1,2,\dots,l,n}(x,t) &\xrightarrow{C(\mathbf{Q})} w_{1,2,\dots,l}(x,t), \\ \partial w_{1,2,\dots,l,n}(x,t)/\partial x &\xrightarrow{L^2(0,1)} \partial w_{1,2,\dots,l}(x,t)/\partial x \quad \text{for every } t \in [0,T], \\ \partial w_{1,2,\dots,l,n}(x,t)/\partial t &\xrightarrow{L^2(0,1)} \partial w_{1,2,\dots,l}(x,t)/\partial t \quad \text{for every } t \in [0,T], \end{aligned}$$

and

$$\partial^2 w_{1,2,\dots,l,n}(x,t)/\partial t\partial x \xrightarrow{L^2(Q)} \partial^2 w_{1,2,\dots,l}(x,t)/\partial t\partial x, \qquad l = 1, 2, 3, \dots, \quad \text{as } n \to \infty.$$

The norms of $w_{1,2,\dots,l}(x,t)$ and their derivatives also satisfy the same estimates as those for the norms $w_{1,2,\dots,l,n}(x,t)$ and their derivatives (3.29). Moreover, it is clear that $w_{1,2,\dots,l}(x,t)$ is the weak solution of (3.12) corresponding to $\{k, h_1, \dots, h_l, f\}$.

Now let the operators B_l be defined as

$$B_l(k) \cdot h_1 h_2 \cdots h_l = w_{1,2,\dots,l}(x,t) \in C(\mathbf{Q})$$

for any $k \in \Sigma$, $h_i \in C[0,T]$, $i = 1, 2, \dots, l$ and $l = 1, 2, 3, \dots$. Obviously, these operators are linear with respect to h_i, $i = 1, 2, \dots, l$.

To prove the Fréchet differentiability of the operator A_1 up to an arbitrary order, one needs to verify the following equalities, one by one:

$$\begin{aligned} &|A_1 \cdot k^* - A_1 \cdot k - B_1(k) \cdot (k^* - k)|_0 = O(|k^* - k|_0), \\ &|A_1'(k^*) \cdot h_1 - A_1'(k) \cdot h_1 - B_2(k) \cdot h_1(k^* - k)|_0 = O(|k^* - k|_0), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &|A_1^{(l-1)}(k^*) \cdot h_1 \cdots h_{l-1} - A_1^{(l-1)}(k) \cdot h_1 \cdots h_{l-1} - B_l(k) \cdot h_1 \cdots h_{l-1}(k^* - k)|_0 \\ &\qquad = O(|k^* - k|_0), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

In fact, upon using the results in Lemma 4 for their approximations, one obtains

$$|u_n^*(x,t) - u_n(x,t) - w_{1n}(x,t)| \le 2^{1/2}(k_- + k_+)k_-^{-3}||f||_{H^1(0,T)}|k^* - k|_0^2$$

and

$$\begin{aligned} &|w_{1,2,\dots,l-1,n}^*(x,t) - w_{1,2,\dots,l-1,n}(x,t) - w_{1,2,\dots,l,n}(x,t)| \\ &\qquad \le 2^{1/2}(l+1)!(k_- + k_+)k_-^{-l-2}||f||_{H^1(0,T)}|h_1|_0 \cdots |h_{l-1}|_0|k^* - k|_0^2, \\ &\qquad\qquad l = 2, 3, \dots. \end{aligned}$$

Hence, as $n \to \infty$, the above equalities hold. By the definition of the Fréchet differential, one has $A_1^{(l)}(k) \cdot h_1 h_2 \cdots h_l = B_l(k) \cdot h_1 h_2 \cdots h_l$ and

$$A^{(l)}(k) \cdot h_1 h_2 \cdots h_l = w_{1,2,\dots,l}(0,t) = -\int_0^1 \partial(w_{1,2,\dots,l}(x,t))/\partial x \, dx \in C[0,T]. \quad \square$$

4. Convergence of the Iterative Solution of a GPST. Before proving the main convergence theorems, we need the following lemma from [12].

LEMMA 7. *Let ϕ be an operator from a Banach space X into another Banach space Y and ϕ be Fréchet differentiable in $\Omega \subset X$. Let N_n be a sequence of linear invertible operators from X into Y such that for given nonnegative constants λ, ξ, β and η,*

(i) $||N_n^{-1}|| \leq \lambda$,

(ii) $||N_n - \phi'(k_0)|| \leq \xi$,

(iii) *if k^*, k are in the sphere $S(k_0, \rho) \subset \Omega$, then*

$$||\phi(k^*) - \phi(k) - \phi'(k_0) \cdot (k^* - k)||_Y \leq \beta ||k^* - k||_X,$$

(iv) $||\phi(k_0)||_Y \leq \eta$, *and*

(v) $\varsigma = \lambda(\beta + \xi) < 1$, $r_0 = \lambda\eta(1 - \varsigma) \leq \rho$.

Then the iteration $k_{n+1} = k_n - N_n^{-1} \cdot \phi(k_n)$, $n = 0, 1, 2, 3, \ldots$, is well defined and converges to a solution $\mathbf{k}$ of $\phi(k) = 0$. Furthermore, $||\mathbf{k} - k_0||_X \leq r_0$, and $\mathbf{k}$ is the only solution contained in this sphere. The rate of convergence is given by $||\mathbf{k} - k_n||_X \leq r_0 \varsigma^n$.

To put the special form of GPST described in Section 2 into the mathematical framework of Lemma 7, we let the operators $\phi(k)$ and N_n be given by (2.5) and (2.6), respectively, and let $X = C[0,1]$, $Y = V_0 = (C[0,1])^*$ and $\Omega = \Sigma$. By using the properties of the operator A proved in the previous section it can be easily shown that $\phi(k)$ is Fréchet differentiable,

$$\phi'(k) = A'^*(k) \cdot A'(k) + A''^*(k) \cdot (A \cdot k - g) + \alpha^2 B,$$

and that there exist positive constants L_1, L_2 and L_3, depending only on k_-, k_+ and $||f||_{H^1(0,T)}$ such that

$$||A'^*(k^*) \cdot (A \cdot k^* - g) - A'^*(k) \cdot (A \cdot k - g)|| \leq L_1 |k^* - k|_0,$$

$$||A'^*(k^*) \cdot A'(k^*) - A'^*(k) \cdot A'(k)|| \leq L_2 |k^* - k|_0$$

and

$$||A''^*(k^*) \cdot (A \cdot k^* - g) - A''^*(k) \cdot (A \cdot k - g)|| \leq L_3 |k^* - k|_0$$

for any $k^*, k \in \Sigma$. Hence Lemma 7 applied to the special form of GPST becomes the following lemma.

LEMMA 8. *Suppose that there exist positive constants λ and η and $k_0 \in \Sigma$ such that*

$$||\phi(k_0)||_Y \leq \eta, \tag{4.1}$$

$$||N_0^{-1}|| = ||\phi'(k_0)^{-1}|| \leq \lambda \tag{4.2}$$

and

$$\lambda^2 \eta \leq (4C)^{-1}, \tag{4.3}$$

where $C = 3L_2 + L_3$, with L_2 and L_3 being the Lipschitz constants of $A'^* \cdot A'$ and $A''^* \cdot (Ak - g)$, respectively. Then the iterative sequence $\{k_n\}$ of (2.6) converges to a solution $k_\alpha(x)$ of $\phi(k) = 0$. Moreover, its rate of convergence is characterized by the estimate

$$||k_\alpha - k_n||_X = r_0 \varsigma^n,$$

where $r_0 = \lambda\eta(1 - C\lambda\rho)^{-1}$ and $\varsigma = \lambda(2L_2 + L_3)\rho(1 - L_2\lambda\rho)^{-1}$.

Proof. To show that the hypotheses (4.1)–(4.3) are equivalent to the hypotheses (i)–(v) of Lemma 7, we note first that (4.1) is equivalent to (iv). Next, let $\rho = (2\lambda C)^{-1}(1-(1-4\lambda^2\eta C)^{1/2})$; obviously, ρ satisfies $\rho = \lambda\eta(1-\lambda\rho C)^{-1}$. From (4.3), $\lambda\rho C < \frac{1}{2}$. Let $S(k_0, \rho) \subset \Sigma$; then for any $k_n \in S(k_0, \rho)$ we have

$$\begin{aligned}||N_n - \phi'(k_0)|| = ||N_n - N_0|| &= ||A'^*(k_n) \cdot A'(k_n) - A'^*(k_0) \cdot A'(k_0)|| \\ &\leq L_2||k_n - k_0||_X \leq L_2\rho = \xi,\end{aligned}$$

which verifies (ii) and $||N_0^{-1}(N_n - N_0)|| \leq L_2\lambda\rho \leq C\lambda\rho < \frac{1}{2}$ from (4.2). From [9], the operator N_n is invertible and $||N_n^{-1}|| \leq \lambda(1 - L_2\lambda\rho)^{-1} = \lambda^*$, which implies (i).

For any $k \in S(k_0, \rho)$, one also has

$$\begin{aligned}||\phi'(k) - \phi'(k_0)|| &\leq ||A'^*(k_n) \cdot A'(k_n) - A'^*(k_0) \cdot A'(k_0)|| \\ &\quad + ||A''^*(k_n) \cdot (A \cdot k_n - g) - A''^*(k_0) \cdot (A \cdot k_0 - g)|| \\ &\leq (L_2 + L_3)\rho;\end{aligned}$$

thus, for any $k^*, k \in S(k_0, \rho)$, one obtains

$$\begin{aligned}&||\phi(k^*) - \phi(k) - \phi'(k_0)(k^* - k)||_Y \\ &= \left\| \int_0^1 \{\phi'(k + t(k^* - k)) - \phi'(k_0)\}\, dt \cdot (k^* - k) \right\|_Y \\ &\leq (L_2 + L_3)||k^* - k||_X = \beta||k^* - k||_X,\end{aligned}$$

which is equivalent to (iii). Finally, let

$$\varsigma = \lambda^*(\beta + \xi) = \lambda(2L_2 + L_3)\rho(1 - L_2\lambda\rho)^{-1}.$$

Then it is clear that $\varsigma < 1$ and $r_0 = \lambda^*\eta(1-\varsigma) \leq \lambda\eta(1 - C\lambda\rho)^{-1} = \rho$, which implies (v). Hence, from Lemma 7, the conclusion of Lemma 8 is proved. □

Now the main result is contained in the following theorem.

THEOREM. *Assume that there exists a regularized solution $k_\alpha(x) \in \Sigma$ for $\alpha > 0$ small enough such that $\phi(k_\alpha) = 0$ and $\phi'(k_\alpha)$ is invertible. Then there exists at least one sequence $\{k_n\}$ from* (2.6) *that converges to $k_\alpha(x)$. Moreover, the following error estimate holds, $||k_\alpha - k_n||_X \leq r_0\varsigma^n$, where r_0 and $\varsigma < 1$ are two positive constants depending only on α, $||B||$, L_1, L_2, L_3 and $||\phi'(k_\alpha)^{-1}||$.*

Proof. To show that there exists a $k_0 \in \Sigma$ such that the hypotheses (4.1)–(4.3) of Lemma 8 are satisfied, let $||\phi'(k_\alpha)^{-1}|| = \frac{1}{2}\lambda$, $\lambda > 0$, and $\delta = \text{Min}\{\lambda^{-1}(L_2 + L_3)^{-1}, \frac{1}{4}\lambda^{-2}C^{-1}(L_1 + \alpha^2||B||)^{-1}\}$. For any $k_0 \in S(k_\alpha, \delta) \cap \Sigma$, one has

$$\begin{aligned}||\phi(k_0)||_Y &= ||\phi(k_0) - \phi(k_\alpha)||_Y \\ &\leq ||A'^*(k_0) \cdot (A \cdot k_0 - g) - A'^*(k_\alpha) \cdot (A \cdot k - g)|| + \alpha^2||B \cdot k_0 - B \cdot k_\alpha|| \\ &\leq (L_1 + \alpha^2||B||)||k_0 - k_\alpha||_X \leq (L_1 + \alpha^2||B||)\delta = \eta,\end{aligned}$$

i.e., (4.1) is satisfied. Since

$$\begin{aligned}||\phi'(k_0) - \phi'(k_\alpha)|| &\leq ||A'^*(k_0) \cdot A'(k_0) - A'^*(k_\alpha) \cdot A'(k_\alpha)|| \\ &\quad + ||A''^*(k_0) \cdot (A \cdot k_0 - g) - A''^*(k_\alpha) \cdot (A \cdot k_\alpha - g)|| \\ &\leq (L_2 + L_3)||k_0 - k_\alpha||_X \leq (L_2 + L_3)\delta \leq \lambda^{-1}\end{aligned}$$

and $||\phi'(k_\alpha)^{-1}|| = \frac{1}{2}\lambda$, one can show that $\phi(k_0)$ is invertible and $||\phi'(k_0)^{-1}|| \leq \lambda$, i.e., (4.2) is satisfied. Next, from (4.1) and (4.2) it follows that $\lambda^2\eta = \lambda^2(L_1 + \alpha^2||B||)\delta \leq \frac{1}{4}C^{-1}$, i.e., (4.3) is satisfied. Finally, Lemma 8 yields the results of the theorem. □

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